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**3 (Sem–4 /CBCS) MAT HC 3
2021**

MATHEMATICS

(Honours)

Paper : MAT–HC–4036

(Ring Theory)

Full Marks : 80

Time : Three hours

***The figures in the margin indicate
full marks for the questions.***

1. 1×6=6
- (a) Give an example of an infinite non-commutative ring with unity.
- (b) What is the characteristic of the ring $Z_3[i]$?
- (c) Find all the idempotent elements of Z_{10} .
- (d) Let $f(x)=4x^3 + 2x^2 + x + 3$ and $g(x)= 3x^4 + 3x^3 + 3x^2 + x + 4$ where $f(x), g(x) \in Z_5[x]$. Compute $f(x).g(x)$.

Contd.

- (e) Show that the polynomial $x^5 + 9x^4 + 12x^2 + 6$ is irreducible over \mathbb{Q} .
- (f) Define an Euclidean Domain.

2. 2×5=10

- (a) Prove that in a ring R , $a(-b) = (-a)b = -(ab)$ for all $a, b \in R$.
- (b) Let A and B be two ideals of a ring R . Show that $AB \subseteq A \cap B$.
- (c) If A is an ideal of a ring R and $1 \in A$, then prove that $A = R$.
- (d) If R is a commutative ring with unity and A is an ideal of R , show that R/A is a commutative ring with unity.
- (e) Let $f(x) = x^3 + 2x + 4$ and $g(x) = 3x + 2$ in $\mathbb{Z}_5[x]$. Determine the quotient and remainder upon dividing $f(x)$ by $g(x)$.

3. Answer **any four** parts : 6×4=24

- (a) Let R be a commutative ring with unity and A be an ideal of R . Prove that R/A is a field if and only if A is a maximal ideal. 6

- (b) (i) Let R be a commutative ring and A be an ideal of it. Show that the set $\{r \in R \mid ra = 0, \text{ for all } a \in A\}$ is an ideal of R . 3
- (ii) Prove that the characteristic of an integral domain is 0 or prime. 3
- (c) Define a principal ideal domain. If F is a field, then show that $F(x)$ is a principal ideal domain. 1+5=6
- (d) Let p be a prime and $f(x) \in Z(x)$ with $\deg f(x) \geq 1$. Suppose $\overline{f(x)}$ be the polynomial in $Z_p[x]$ obtained from $f(x)$ by reducing all the coefficients of $f(x)$ modulo p . If $\overline{f(x)}$ is irreducible over Z_p and $\deg f(x) = \deg \overline{f(x)}$, then prove that $f(x)$ is irreducible over Q .
Is the converse true? Justify your answer. 4+2=6
- (e) Prove that in a principal ideal domain, an element is an irreducible if and only if it is a prime. 6
- (f) (i) Prove that the ring of integers Z is an Euclidean Domain. 2

(ii) Prove that every Euclidean Domain is a principal ideal domain. 4

4. Answer **any four** parts : 10×4=40

(a) (i) Let $Z[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in Z\}$.

Prove that $Z(\sqrt{2})$ is a ring under the ordinary addition and multiplication of real numbers. 6

(ii) Let R be a ring. Prove that $a^2 - b^2 = (a + b)(a - b)$ for all a, b in R if and only if R is commutative. 4

(b) (i) Define a field. Prove that a finite integral domain is a field. Hence show that for any prime p , Z_p the ring of integers modulo p , is a field. 1+5+2=8

(ii) Show that 0 is the only nilpotent element in an integral domain. 2

(c) (i) Show that $S = \{a + ib \mid a, b \in Z, b \text{ is even}\}$ is a subring of $Z[i]$ but not an ideal of $Z[i]$. 4

(ii) Prove that the only ideals of a field F are $\{0\}$ and F itself. 2

(iii) Show that $\frac{R[x]}{\langle x^2+1 \rangle}$ is a field. 4

(d) (i) Let ϕ be a monomorphism from a ring R to a ring S .
Prove that kernel of ϕ is an ideal of R . 4

(ii) Let ϕ be a homomorphism from a ring R to a ring S . Prove that $R/\text{Ker } \phi \cong \phi(R)$. 6

(e) (i) If ϕ is an isomorphism from a ring R to a ring S , then show that ϕ^{-1} is an isomorphism from S to R . 5

(ii) Let R be a ring with unity e . Prove that the mapping $\phi : Z \rightarrow R$ given by $n \rightarrow ne$ is a ring homomorphism. 5

(f) Let F be a field and let $f(x)$ and $g(x) \in F[x]$ with $g(x) \neq 0$. Prove that there exist unique polynomials $q(x)$ and $r(x)$ in $F(x)$ such that $f(x) = g(x)q(x) + r(x)$ and either $r(x) = 0$ or $\deg r(x) < \deg g(x)$. Hence show that if $a \in F$ and $f(x) \in F[x]$, then $f(a)$ is the remainder in the division of $f(x)$ by $x - a$.

7+3=10
