

Total number of printed pages-11

3 (Sem-4/CBCS) MAT HC 3

2022

**MATHEMATICS**

(Honours)

Paper : MAT-HC-4036

**(Ring Theory)**

Full Marks : 80

Time : Three hours

***The figures in the margin indicate full marks for the questions.***

1. Answer **any ten** : 1×10=10

(a) The set  $Z$  of integers under ordinary addition and multiplication is a commutative ring with unity 1. What are the units of  $Z$ ?

(b) What is the trivial subring of  $R$ ?

Contd.

- (c) What are the elements of  $Z_3[i]$ ?
- (d) Give the definition of zero divisor.
- (e) Give an example of a commutative ring without zero divisors that is not an integral domain.
- (f) What is the characteristic of an integral domain?
- (g) Why is the idea  $\langle x^2 + 1 \rangle$  not prime in  $Z_2[x]$ ?
- (h) Find all maximal ideals in  $Z_8$ .
- (i) Is the mapping from  $Z_5$  to  $Z_{30}$  given by  $x \rightarrow 6x$  is a ring homomorphism?

(j) If  $\phi$  is an isomorphism from a ring  $R$  onto a ring  $S$ , then  $\phi^{-1}$  is an isomorphism from  $S$  onto  $R$ .  
Write True or False.

(k) Is the ring  $2z$  isomorphic to the ring  $3z$ ?

(l) Let  $f(x) = x^3 + 2x + 4$  and  $g(x) = 3x + 2$  is  $z_5[x]$ . Determine the quotient and remainder upon dividing  $f(x)$  by  $g(x)$ .

(m) Why is the polynomial  $3x^5 + 15x^4 - 20x^3 + 10x + 20$  irreducible over  $Q$ ?

(n) Give the definition of Euclidean domain.

(o) State the second isomorphism theorem for rings.

2. Answer **any five** :  $2 \times 5 = 10$

(a) Define ring. What is the unity of a polynomial ring  $Z[x]$ ?

(b) Prove that in a ring  $R$ ,  $(-a)(-b) = ab$  for all  $a, b \in R$ .

(c) Prove that set  $S$  of all matrices of the form  $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$  with  $a$  and  $b$ , forms a sub-ring of the ring  $R$  of all  $2 \times 2$  matrices having elements as integers.

(d) Let  $R$  be a ring with unity 1. If 1 has infinite order under addition, then the characteristic of  $R$  is 0. If 1 has order  $n$  under addition, then prove that the characteristic of  $R$  is  $n$ .

(e) Let  $Z/4Z = \{0 + 4z, 1 + 4z, 2 + 4z, 3 + 4z\}$ .  
Find  $(2 + 4z) + (3 + 4z)$  and  $(2 + 4z)(3 + 4z)$ .

(f) Let  $R = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \mid a, b \in Z \right\}$  and let  $\phi$  be

the mapping defined as  $\begin{bmatrix} a & b \\ b & a \end{bmatrix} \rightarrow a - b$ .

Show that  $\phi$  is a homomorphism.

(g) Let  $f(x) = 4x^3 + 2x^2 + x + 3$  and

$$g(x) = 3x^4 + 3x^3 + 3x^2 + x + 4$$

where  $f(x), g(x) \in Z_5[x]$ .

Compute  $f(x) + g(x)$  and  $f(x) \cdot g(x)$ .

(h) Prove that in an integral domain, every prime is an irreducible.

3. Answer **any four** :  $5 \times 4 = 20$

(a) Define a sub-ring. Prove that a non-empty subset  $S$  of a ring  $R$  is a sub-ring if  $S$  is closed under subtraction and multiplication, that is if  $a - b$  and  $ab$  are in  $S$  whenever  $a$  and  $b$  are in  $S$ .  
 $1 + 4 = 5$

(b) Prove that the ring of Gaussian integers

$Z[i] = \{a + ib \mid a, b \in Z\}$  is an integral domain.

(c) Let  $R$  be a commutative ring with unity and let  $A$  be an ideal of  $R$ . Then prove that  $R/A$  is an integral domain if and only if  $A$  is prime.

(d) If  $D$  is an integral domain, then prove that  $D[x]$  is an integral domain.

(e) (i) If  $R$  is commutative ring then prove that  $\phi(R)$  is commutative, where  $\phi$  is an isomorphism on  $R$ . 3

(ii) If the ring  $R$  has a unity 1,  $S \neq \{0\}$  and  $\phi: R \rightarrow S$  is onto, then prove that  $\phi(1)$  is the unity of  $S$ . 2

(f) Let  $f(x) \in Z[x]$ . If  $f(x)$  is reducible over  $Q$ , then prove that it is reducible over  $Z$ .

(g) Consider the ring

$S = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \mid a, b \in Z \right\}$ . Show that

$\phi: \mathbb{C} \rightarrow S$  is given by

$\phi(a + bi) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$  is a ring

isomorphism.

(h) Prove that  $Z[i] = \{a + bi \mid a, b \in Z\}$ , the ring of Gaussian integers is an Euclidean domain.

4. Answer **any four** : 10×4=40

(a) (i) Prove that the set of all continuous real-valued functions of a real variable whose graphs pass through the point  $(1, 0)$  is a commutative ring without unity under the operation of pointwise addition and multiplication [that is, the operations  $(f + g)(a) = f(a) + g(a)$  and  $(f \cdot g)(a) = f(a) \cdot g(a)$ . 6

(ii) Prove that if a ring has a unity, it is unique and if a ring element has an inverse, it is unique. 4

(b) Define a field. Is the set  $I$  of all integers a field with respect to ordinary addition and multiplication? Let

$Q[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in Q\}$ . Prove that

$Q[\sqrt{2}]$  is a field. 2+1+7=10

(c) (i) Prove that the intersection of any collection of subrings of a ring  $R$  is a sub-ring of  $R$ . 5

(ii) Let  $R$  be a commutative ring with unity and let  $A$  be an ideal of  $R$ . Prove that  $R/A$  is a field if  $A$  is maximal. 5

(d) Define factor ring. Let  $R$  be a ring and let  $A$  be a subring of  $R$ . Prove that the set of co-sets  $\{r + A \mid r \in R\}$  is a ring under the operation

$$(s + A) + (t + A) = (s + t) + A \text{ and}$$

$(s + A)(t + A) = st + A$  if and only if  $A$  is an ideal of  $R$ . 1+5+4=10

(e) (i) Let  $\phi$  be a ring homomorphism from  $R$  to  $S$ . Prove that the mapping from  $R/\ker \phi$  to  $\phi(R)$ , given by  $r + \ker \phi \rightarrow \phi(r)$  is an isomorphism. 5

(ii) Let  $R$  be a ring with unity and the characteristic of  $R$  is  $n > 0$ . Prove that  $R$  contains a sub-ring isomorphic to  $Z_n$ . If the characteristic of  $R$  is 0, then prove that  $R$  contains a sub-ring isomorphic to  $Z$ . 3+2=5

(f) Let  $F$  be a field and let  $p(x) \in F[x]$ . Prove that  $\langle p(x) \rangle$  is a maximal ideal in  $F[x]$  if and only if  $p(x)$  is irreducible over  $F$ .

(g) Let  $F$  be a field and let  $f(x)$  and  $g(x) \in F[x]$  with  $g(x) \neq 0$ . Prove that there exists unique polynomials  $q(x)$  and  $r(x)$  in  $F[x]$  such that  $f(x) = g(x)q(x) + r(x)$  and either  $r(x) = 0$  or  $\text{degr}(x) < \text{deg } g(x)$ . With the help of an example verify the division algorithm for  $F[x]$ . 7+3=10

(h) (i) If  $F$  is a field, then prove that  $F[x]$  is a principal ideal domain. 5

(ii) Let  $F$  be a field and let  $p(x), a(x), b(x) \in F[x]$ . If  $p(x)$  is irreducible over  $F$  and  $p(x) | a(x)b(x)$ , then prove that  $p(x) | a(x)$  or  $p(x) | b(x)$ .

5

(i) Prove that every principal ideal domain is a unique factorization domain.

(j) (i) Prove that every Euclidean domain is a principal ideal domain. 5

(ii) Show that the ring

$$\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$$

is an integral domain but not a unique factorization domain. 5