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3 (Sem–4/CBCS) MAT HC 1

2022

MATHEMATICS

(Honours)

Paper : MAT–HC–4016

(Multivariate Calculus)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

1. Answer ***any ten*** : $1\times10=10$

(i) Find the domain of $f(x, y) = \dfrac{1}{\sqrt{x-y}}$.

(ii) How is directional derivative of a function at a point related to the gradient of the function at that point ?

(iii) Define harmonic function ?

(iv) Define $\iint\limits_R f(x, y)\, dA$.

Contd.

(v) Write the value of $\vec{\nabla}(f^n)$.

(vi) Define critical point.

(vii) Define relative extrema for a function of two variables.

(viii) When is a curve said to be positively oriented?

(ix) Describe the fundamental theorem of line integral.

(x) When is a surface said to be smooth?

(xi) Compute $\int_1^4 \int_{-2}^3 \int_2^5 dx\,dy\,dz$.

(xii) Evaluate $\underset{(x,y)\to(1,3)}{Lt} \frac{x-y}{x+y}$.

(xiii) If $f(x, y) = x^3y + x^2y^2$, find f_x.

(xiv) When is a line integral said to be path independent?

(xv) Explain the difference between $\int_C f\,ds$ and $\int_C f\,dx$.

2. Answer **any five** questions : 2×5=10

(a) Sketch the level surface $f(x, y, z) = c$ if $(x, y, z) = y^2 + z^2$ for $c = 1$.

(b) Determine f_x and f_y for $f(x, y) = xy^2 \ln(x+y)$.

(c) Find $\frac{\partial w}{\partial t}$ if $w = \ln\left(x + 2y - z^2\right)$ and $x = 2t - 1$, $y = \frac{1}{t}$, $z = \sqrt{t}$.

(d) Evaluate $\int_1^2 \int_0^{\pi} x \cos y\,dy\,dx$.

(e) Evaluate $\underset{(x,y)\to(0,0)}{Lim} \frac{x^2 + x - xy - y}{x - y}$.

(f) Define line integral over a smooth curve.

(g) Find the Jacobian $\frac{\partial(x, y)}{\partial(u, v)}$ when $x = u + 2v$, $y = 3u - 4v$.

(h) Using polar coordinates find the limit $\underset{(x,y)\to(0,0)}{Lim} \frac{\tan\left(x^2 + y^2\right)}{x^2 + y^2}$.

3. Answer ***any four*** : 5×4=20

(a) Describe the graph of the function $f(x, y) = 1 - x - \frac{1}{2}y$.

(b) Use the method of Lagrange's multipliers to find the maximum and minimum values of $f(x, y) = 1 - x^2 - y^2$ subject to the constraints $x + y = 1$ with $x \geq 0$, $y \geq 0$.

(c) Evaluate $\int_C [(y - x)dx + x^2 y\, dy]$, where C is the curve defined by $y^2 = x^3$ from $(1, -1)$ to $(1, 1)$.

(d) Examine the continuity of the following function at the origin :

$$f(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

(e) Find $\frac{\partial w}{\partial s}$ if $w = 4x + y^2 + z^3$ where $x = e^{rs^2}$, $y = \ln\frac{r+s}{t}$ and $z = rst^2$.

(f) Suppose the function f is differentiable at the point P_0 and that the gradient at P_0 satisfies $\Delta f_0 \neq 0$. Show that Δf_0 is orthogonal to the level surface of f through P_0.

(g) Compute $\iint_D \left(\frac{x - y}{x + y}\right)^4 dy\,dx$ where D is the triangular region bounded by the line $x + y = 1$ and the coordinate axes, using change of variables $u = x - y$, $v = x + y$.

(h) Find the absolute extrema of $f(x, y) = 2x^2 - y^2$ on the closed bounded set S, where S is the disk $x^2 + y^2 \leq 1$.

4. Answer ***any four*** questions : 10×4=40

(a) The radius and height of a right circular cone are measured with errors of at most 3% and 2% respectively. Use increments to approximate the maximum possible percentage error in computing the volume of the cone using these measurements and the formula $V = \frac{1}{3}\pi R^2 H$.

(b) Let $f(x, y) = \begin{cases} xy\left(\dfrac{x^2 - y^2}{x^2 + y^2}\right), & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

Show that $f_x(0, y) = -y$ and $f_x(x, 0) = x$ for all x and y. Then show that $f_{xy}(0, 0) = -1$ and $f_{yx}(0, 0) = 1$.

Contd.

(c) (i) Find the directional derivative of $f(x, y) = ln\left(x^2 + y^3\right)$ at $P_0(1, -3)$ in the direction of $\vec{v} = 2i - 3j$.

(ii) In what direction is the function defined by $f(x, y) = xe^{2y-x}$ increasing most rapidly at the point $P_0(2, 1)$, and what is the maximum rate of increase? In what direction is f decreasing most rapidly?

(d) When two resistances R_1 and R_2 are connected in parallel, the total resistance R satisfies $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$. If R_1 is measured as 300 *ohms* with maximum error of 2% and R_2 is measured as 500 *ohms* with a maximum error of 3%, what is the maximum percentage error in R ?

(e) Verify the vector field $\vec{F} = \left(e^x \sin y - y\right)i + \left(e^x \cos y - x - 2\right)j$ is conservative. Also find the scalar potential function f for $\vec{F}$.

(f) (i) Evaluate $\iiint_D \frac{dxdydz}{\sqrt{x^2 + y^2 + z^2}}$ where D is the solid sphere $x^2 + y^2 + z^2 \le 3$.

(ii) Find the volume of the solid D, where D is bounded by the paraboloid $z = 1 - 4\left(x^2 + y^2\right)$ the xy-plane.

(g) (i) Use a polar double integral to show that a sphere of radius a has volume $\frac{4}{3}\pi a^3$.

(ii) Evaluate $\int_0^3 \int_0^{\sqrt{9-x^2}} xdydx$ by converting to polar coordinates.

(h) State Green's theorem. Verify Green's theorem for the line integral $\oint_C \left(y^2dx + x^2dy\right)$ where C is the square having vertices (0, 0), (1, 0), (1, 1) and (0, 1).

(i) State Stokes' theorem. Using Stokes' theorem evaluate the line integral $\oint_C \left(x^3y^2dx + dy + z^2dz\right)$, where C is the circle $x^2 + y^2 = 1$ and in the plane $z = 1$, counterclockwise when viewed from the origin.

(j) A container in R^3 has the shape of the cube given by $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$. A plate is placed in the container in such a way that it occupies that portion of the plane $x + y + z = 1$ that lies in the cubical container. If the container is heated so that the temperature at each point (x, y, z) is given by $T(x, y, z) = 4 - 2x^2 - y^2 - z^2$ in hundreds of degrees Celsius, what are the hottest and coldest points on the plate? You may assume these extreme temperatures exist.
