2017

MATHEMATICS

(Major)

Paper : 5.2

(Topology)

Full Marks : 60

Time : 3 hours

The figures in the margin indicate full marks for the questions

1. Answer the following questions : 1×7=7

(a) Find the derived set of the sets $A_1 =]0, 1[$ and $A_2 = [0, 1]$ in the real line $\mathbb{R}$ with the usual metric.

(b) Give an example to show that the intersection of an infinite family of open sets need not be open in a metric space.

(c) State Cantor's intersection theorem for metric spaces.

(d) Define the usual metric on $\mathbb{R}$.

(e) Give an example to show that $\overline{A \cap B} \neq \bar{A} \cap \bar{B}$, in a topological space.

(f) Let

$$X = \{1, 2, 3, 4\} \text{ and}$$
$$\mathscr{T} = \{\phi, X, \{1\}, \{2\}, \{1, 2\}, \{2, 3, 4\}\}$$

Let $f : X \to X$ be defined by $f(1) = 2$, $f(2) = 4$, $f(3) = 2$, $f(4) = 3$. State whether f is continuous at 3 or not.

(g) Define a norm on the set $\mathbb{C}^n$.

2. Answer the following questions : 2×4=8

(a) In a metric space, every convergent sequence is a Cauchy sequence. Justify whether it is true or false.

(b) In the cofinite topological space $(X, \mathscr{T})$, find the closure of any subset A of X.

(c) Show that every inner product space is a normed linear space.

(d) Show that in a normed linear space $(X, \|\cdot\|)$, $|\,\|x\| - \|y\|\,| \leq \|x - y\| \; \forall \; x, y \in X$.

3. Answer the following questions : 5×3=15

(a) Let (X, d) be a metric space. If x_0 is a limit point of a subset A of X, then prove that there exists a sequence $\{a_n\}$ of points of A, all distinct from x_0, which converges to x_0. 5

(b) Let $(X, \mathscr{T})$ be a topological space and Y be a nonempty subset of X. Prove that $u = \{G \cap Y : G \in T\}$ will be a topology on Y. Give the name of this topology. 4+1

Or

Let X and Y be topological spaces and f be a bijective mapping of X to Y. Prove that f is continuous and open if and only if it is a homeomorphism. 5

(c) If x and y are any two vectors in an inner product space $(X, \langle\cdot, \cdot\rangle)$, then prove that $|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$. 5

Or

In an inner product space $(X, \langle\cdot, \cdot\rangle)$, if $x_n \to x$ and $y_n \to y$, then show that $\langle x_n, y_n \rangle \to \langle x, y \rangle$. 5

4. Answer the following questions : 10×3=30

(a) Let $C[a, b]$ denote the set of all real valued continuous functions defined on $[a, b]$. Prove that $C[a, b]$ is complete with respect to a suitable metric defined on it.

Or

Let (X, d) be a metric space and A be a subset of X. Prove that—

(i) A is closed if and only if A contains all its limit points;

(ii) a point $x \in X$ is a limit point of A if and only if every open sphere centred at x contains infinitely many points of A.

(b) State and prove Baire's category theorem for metric spaces.

Or

Prove that a metric space is second countable if and only if it is separable.

(c) Prove that a metric space is compact if and only if every collection of closed subsets of X with the finite intersection property has a nonempty intersection.

Or

Prove that a subspace of the real line $\mathbb{R}$ is connected if and only if it is an interval.

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