2012

MATHEMATICS

(Major)

Paper : 3.2

(Linear Algebra and Vector)

Full Marks : 80

Time : 3 hours

The figures in the margin indicate full marks for the questions

GROUP—A

(Linear Algebra)

(*Marks* : 40)

1. Answer the following : 1×6=6

(a) Is the following statement true or false? If false, correct the statement :

For linearly independent vectors v_1, v_2, v_3 in a vector space V the set $\{v_1, v_3\}$ is a linearly dependent set.

(b) What is the basis of the vector space $V = \{0_v\}$?

(c) State the condition under which a set of m vectors spans $\mathbb{R}^n$.

(d) What are the eigenvalues of an upper triangular matrix?

(e) State Cayley-Hamilton theorem.

(f) Let $T : V \to V$ be a linear operator. State one condition on T so that 0 is an eigenvalue of T.

2. Answer the following : 2×2=4

(a) Consider the vector space $V = \mathbb{R}^3$ over $\mathbb{R}$. If U and W are the xy-plane and yz-plane respectively, then determine $\dim(U \cap W)$.

(b) Find all eigenvalues of the operator

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

defined by

$$T(x, y) = (3x + 3y, x + 5y)$$

3. Answer any *one* part : 10

(a) (i) Let V be the vector space of all functions from the real field $\mathbb{R}$ into $\mathbb{R}$. Show that W is a subspace of V, where

$$W = \{f : f(7) = f(1)\}$$

(ii) Let V be a finite dimensional vector space. Prove that every basis of V has the same number of vectors.

(iii) Determine whether or not the following set S forms a basis of $\mathbb{R}^3$:

$$S = \{(2, 4, -3), (0, 1, 1), (0, 1, -1)\}$$

3+4+3=10

(b) (i) Prove that the intersection of a finite collection of subspaces of a vector space $V(F)$ is a subspace of $V(F)$. Is it true for the union of subspaces?

(ii) Let $V(F)$ be a vector space of dimension n. Prove that any $n+1$ vectors of V are linearly dependent. Further prove that if vectors $v_1, v_2, \cdots, v_n$ span V, then they are linearly independent.

(iii) Find a basis and dimension of the subspace U of $\mathbb{R}^4$, where

$$U = \{(a, b, c, d) \mid a + b = 0, c = 2d\}$$

3+4+3=10

4. Answer any *two* parts : 5×2=10

(a) (i) Let V be the vector space of $n \times n$ square matrices over the field K and M be an arbitrary matrix in V. Show that the map $T : V \to V$ defined by $T(A) = AM + MA, \ \forall A \in V$ is linear.

(ii) Let $T:\mathbb{R}^2 \to \mathbb{R}$ be the linear mapping for which $T(1, 1) = 3$ and $T(0, 1) = -2$. Then find $T(x, y)$. 2+3=5

(b) Verify the rank nullity theorem for the linear mapping $T:\mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T(x, y, z) = (x + y, y + z)$$ 5

(c) Let U and V be vector spaces over a field K. If $\dim U = m$, $\dim V = n$, then prove that $\dim \mathrm{Hom}(U, V) = mn$, where $\mathrm{Hom}(U, V)$ denotes the vector space of all linear mappings from U into V. 5

5. Answer any *one* part : 10

(a) *(i)* Find the eigenvalues and the corresponding eigenvectors of the following matrix :

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$$

(ii) Find the minimal polynomial of the matrix

$$A = \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}$$

Are the characteristic polynomial of A and the minimal polynomial of A same? 5+5=10

(b) (i) Show that the following system of linear equations are consistent. Hence solve them :

$$x+2y-z=3$$
$$3x-y+2z=1$$
$$2x-2y+3z=2$$
$$x-y+z=-1$$

(ii) Prove that the minimal polynomial of a matrix A divides every polynomial which has A as a zero.

(iii) Use Cayley-Hamilton theorem to find the inverse of the matrix

$$\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$$

5+3+2=10

GROUP—B

(Vector)

(*Marks* : 40)

6. Answer the following : 1×4=4

(a) Write the geometrical interpretation of scalar triple product $\vec{a}\cdot(\vec{b}\times\vec{c})$.

(b) Does associative law for cross products of vectors hold?

(c) Write the condition for a vector function $\vec{f}$ of a scalar variable t to be of constant magnitude.

(d) Find $\text{div}\,\vec{r}$, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

7. Answer the following : 2×3=6

(a) A particle moves along the curve $x = 3t^2$, $y = t^2 - 2t$, $z = t^3$, where t is the time. Find the component of velocity at time $t = 1$ in the direction $\hat{i} + \hat{j} - \hat{k}$.

(b) Show that the vector

$$\vec{v} = yz\hat{i} + zx\hat{j} + xy\hat{k}$$

is irrotational.

(c) Evaluate

$$\iint_S \vec{r} \cdot \hat{n}\,dS$$

where S is a closed surface.
(Symbols with usual meanings.)

8. Answer any *one* part : 10

(a) (i) Prove that

$$[\vec{b} \times \vec{c} \;\; \vec{c} \times \vec{a} \;\; \vec{a} \times \vec{b}] = [\vec{a}\,\vec{b}\,\vec{c}]^2$$

(ii) Show that grad ϕ is a vector perpendicular to the surface $\phi(x, y, z) = c$, where c is a constant. Further show that curl (grad ϕ) $= \vec{0}$.

(iii) Given

$$\vec{r}(t) = \hat{i} - 2\hat{j} + 2\hat{k} \quad \text{at} \quad t = 2$$
$$= 2\hat{i} - \hat{j} + 4\hat{k} \quad \text{at} \quad t = 3$$

then evaluate

$$\int_2^3 \left(\vec{r} \cdot \frac{d\vec{r}}{dt} \right) dt$$

3+4+3=10

(b) (i) Prove that

$$[\vec{a} + \vec{b} \quad \vec{b} + \vec{c} \quad \vec{c} + \vec{a}] = 2[\vec{a}\,\vec{b}\,\vec{c}]$$

(ii) Find the angle between the surfaces

$x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$

at the point $(2, -1, 2)$.

(iii) If $\vec{F} = 3xy\hat{i} - y^2\hat{j}$, then evaluate

$$\int_C \vec{F} \cdot d\vec{r}$$

where C is the curve $y = 2x^2$ in the xy-plane from $(0, 0)$ to $(1, 2)$.

3+4+3=10

9. Answer any *two* parts : 5×2=10

(a) (i) If

$$\frac{d\vec{r}}{dt} = \vec{w} \times \vec{r}$$

and

$$\frac{d\vec{s}}{dt} = \vec{w} \times \vec{s}$$

show that

$$\frac{d}{dt}(\vec{r} \times \vec{s}) = \vec{w} \times (\vec{r} \times \vec{s})$$

(ii) If

$$\vec{A} = \cos xy\hat{i} + (3xy - 2x^2)\hat{j} + (3x + 2y)\hat{k}$$

then find

$$\frac{\partial^2 A}{\partial x \partial y}$$

3+2=5

(b) If $\vec{a}$ is a constant vector and

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

show that—

(i) $\text{div}(\vec{a} \times \vec{r}) = 0$

(ii) $\text{curl}(\vec{a} \times \vec{r}) = 2\vec{a}$ 2+3=5

(c) Prove that

$$\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$$

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10. Answer any *one* part : 10

(a) *(i)* Find the value of x so that the vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}+2\hat{j}+3\hat{k}$ and $3\hat{i}+x\hat{j}+5\hat{k}$ are coplanar.

(ii) Let $\vec{a}$, $\vec{b}$, $\vec{c}$ be three unit vectors such that

$$\vec{a}\times(\vec{b}\times\vec{c})=\frac{1}{2}\vec{b}$$

If $\vec{b}$ and $\vec{c}$ are non-parallel vectors, then find the angles which $\vec{a}$ makes with $\vec{b}$ and $\vec{c}$.

(iii) If $\vec{F}=(2x^2-3z)\hat{i}-2xy\hat{j}-4x\hat{k}$, then evaluate

$$\int_V \operatorname{div}\vec{F}\cdot dV$$

where V is the closed region bounded by the planes $x=0$, $y=0$, $z=0$ and $2x+2y+z=4$. 2+3+5=10

(b) *(i)* Prove that

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=[\vec{a}\,\vec{b}\,\vec{d}]\vec{c}-[\vec{a}\,\vec{b}\,\vec{c}]\vec{d}$$

$$=[\vec{a}\,\vec{c}\,\vec{d}]\vec{b}-[\vec{b}\,\vec{c}\,\vec{d}]\vec{a}$$

Hence express any vector $\vec{r}$ in terms of $\vec{a}$, $\vec{b}$, $\vec{c}$ provided they are not coplanar.

(ii) Evaluate

$$\iint_S \vec{F} \cdot \hat{n} dS$$

where

$$\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$$

and S is that part of the surface of the sphere $x^2 + y^2 + z^2 = 1$ which lies in the first octant. 5+5=10

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